Stability of a hard-core fluid jet of small electrical conductivity

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The stability of a hard-core fluid jet is investigated under the assumption of small electrical conductivity of the fluid for parallel and antiparallel current systems. For axisymmetric deformations the parallel current has a destabilizing influence whereas the antiparallel current improves the stability of the configuration. However, for non-axisymmetric deformations characterized by $m = 1$, it is shown that the jet becomes unstable in general, both for parallel and antiparallel current systems. The presence of an external conductor has been shown to have a stabilizing influence on kink instability.

1. Introduction

The problem of stability of the hard-core fluid jet is of considerable interest in plasma confinement and has been investigated in recent years (Anderson, Baker, Ise, Kunkel, Pyle & Stone 1958; Anderson, Furth, Stone & Wright 1958; Colgate & Furth 1959, 1960; Jukes 1961; Reynolds *et al.* 1959). It has been shown on theoretical grounds that with the assumption of ideal conductivity of the fluid, the hard-core plasma jet is stable when the currents are antiparallel and the total current in the jet is less than that in the core. However, experiments have revealed it to be unstable at sufficiently high current densities. In an earlier paper Tandon & Talwar (1961) investigated the stability of the hard-core model of the pinch under the other extreme condition, i.e. vanishingly small electrical conductivity, the fluid being incompressible and inviscid. They found that a current in the core in the same direction as that in the liquid has, in general, a destabilizing influence, whereas an antiparallel current improves the stability of the configuration, a result not arrived at by Lehnert & Sjögren (1960) in their experimental investigation with liquid mercury. The above investigation (Tandon & Talwar), based on energy arguments, was restricted to include only sausage type ($m = 0$) disturbances. The present note is concerned with a generalized study so as to include non-axisymmetric disturbances and also the effect of an external conductor. It is found that the configuration is, in general, unstable both for parallel and antiparallel current systems for kink-type disturbances ($m = 1$). The external conductor has no influence on sausage perturbations ($m = 0$) but has a stabilizing influence on kink instability for long-wave perturbations.

2. Formulation of the problem

The model adopted is as shown in figure 1, where the fluid is confined between a metallic core of radius R_0 and a rigid perfectly conducting wall of radius R_2. The core (insulated from the fluid by a thin sheath) carries a total axial current I_c and the fluid a corresponding current I which can be reversed in axial direction *independently* of the current in the core. The corresponding current densities are denoted by j_0 $(= I_c/\pi R_0^2)$ and j_1 $(= I/\pi(R_1^2 - R_0^2))$, respectively, where R_1 is the outer radius of the fluid jet.

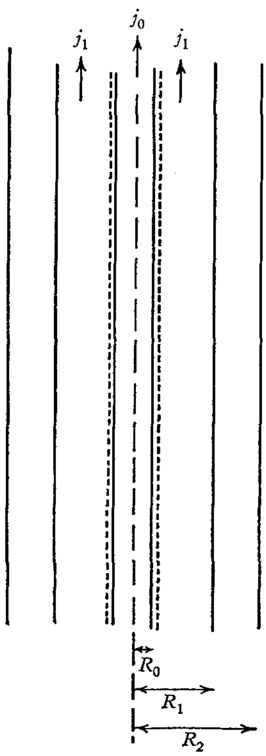

FIGURE 1. The metallic core (radius R_0) is insulated by a thin sheath (dotted line) from the fluid confined to the region R_0 to R_1. R_2 represents the radius of the external conducting wall. j_0 and j_1 denote the current densities in the core and the fluid jet, respectively.

Assuming the fluid to be inviscid and incompressible, the basic equations can be written as

$$\rho\, \partial \mathbf{u}/\partial t + \rho(\mathbf{u}\,.\nabla)\,\mathbf{u} = -\nabla p + \mathbf{j} \times \mathbf{H}, \tag{1}$$

$$\nabla . \mathbf{u} = 0, \tag{2}$$

$$\nabla . \mathbf{H} = 0, \tag{3}$$

$$\nabla \times \mathbf{H} = 4\pi \mathbf{j} \tag{4}$$

and

$$\partial \mathbf{H}/\partial t = \nabla \times (\mathbf{u} \times \mathbf{H}) + (1/4\pi\sigma)\,\nabla^2 \mathbf{H}. \tag{5}$$

Equation (5), in the limit of vanishingly small conductivity, reduces to

$$\nabla^2 \mathbf{H} = 0. \tag{6}$$

The equilibrium configuration is defined by the equations

$$\nabla p_0 = \mathbf{j}_1 \times \mathbf{H}_1, \tag{7}$$

$$\nabla \times \mathbf{H} = 4\pi \mathbf{j}_1 \tag{8}$$

and

$$\nabla . \mathbf{H} = 0, \tag{9}$$

and is characterized by the following expressions for magnetic field:

$$\mathbf{H}_0 = (0, 2\pi j_0 r, 0) \qquad (r \leqslant R_0); \tag{10a}$$

$$\mathbf{H}_1 = (0, 2\pi r^{-1}[j_1 r^2 + R_0^2(j_0 - j_1)], 0) \qquad (R_0 \leqslant r \leqslant R); \tag{10b}$$

$$\mathbf{H}_2 = (0, 2\pi r^{-1}[j_1 R^2 + R_0^2(j_0 - j_1)], 0) \qquad (r \geqslant R). \tag{10c}$$

It follows from equation (7) that the fluid jet has a tendency to detach itself from the core in the antiparallel current system. This tendency is assumed to be arrested by an appropriate externally applied gas pressure. The hydrodynamics of this medium is not taken into account since it does not involve any new physical feature in the stability criterion in our present context (see Appendix).

In order to investigate the stability of the static situation depicted in figure 1, we impart a small perturbation of the first order of smallness, the result of which is to change the free boundary of the fluid into the form

$$r = R + a(t)\cos\phi, \tag{11}$$

where $\phi = -m\theta + kz$ and $a(t)$ is of the form $a_0 e^{i\omega t}$. Conservation of mass per unit length of the cylinder leads to

$$R_1^2 = R^2 + \tfrac{1}{2}a^2. \tag{12}$$

It should be noted that in the approximation considered here the difference between R_1 and R can be ignored. Let the corresponding changes in current density, magnetic field and pressure be denoted by $\delta\mathbf{j}$, $\mathbf{h}^{(i)}$ and δp respectively inside the fluid and $\mathbf{h}^{(e)}$ external to the fluid. We shall calculate the expressions for these perturbations in terms of the amplitude a of the boundary displacement. With the help of suitable boundary conditions we derive finally in § 5 an expression for the dispersion relation.

3. Perturbation in current density and magnetic field

For a fluid of vanishingly small conductivity, from equation (6), it follows that

$$\nabla^2 \mathbf{h}^{(i)} = 0, \tag{13}$$

so that $\delta\mathbf{j}$ is irrotational and derivable from a scalar function ψ. Assuming no accumulation of charge it follows that

$$\nabla^2 \psi = 0. \tag{14}$$

Solution of equation (14) gives

$$\psi = [C_1 I_m(kr) + C_2 K_m(kr)]\sin\phi \tag{15}$$

and

$$\delta j_r = k[C_1 I_m'(kr) + C_2 K_m'(kr)]\sin\phi, \tag{16a}$$

$$\delta j_\theta = -mr^{-1}[C_1 I_m(kr) + C_2 K_m(kr)]\cos\phi, \tag{16b}$$

$$\delta j_z = k[C_1 I_m(kr) + C_2 K_m(kr)]\cos\phi. \tag{16c}$$

Here I_m and K_m are modified Bessel functions of the first and second kind and primes denote their first derivatives with respect to kr. Coupling of equations (16) with

$$\nabla \times \mathbf{h}^{(i)} = 4\pi\delta\mathbf{j} \tag{17}$$

leads to the following expression for the components of $\mathbf{h}^{(i)}$:

$$h_r^{(i)} = [-(4\pi m/kr)\{C_1 I_m(kr) + C_2 K_m(kr)\} + C_3 I'_m(kr) + C_4 K'_m(kr)] \sin\phi, \tag{18a}$$

$$h_\theta^{(i)} = [4\pi\{C_1 I'_m(kr) + C_2 K'_m(kr)\} - (m/kr)\{C_3 I_m(kr) + C_4 K_m(kr)\}] \cos\phi, \tag{18b}$$

$$h_z^{(i)} = [C_3 I_m(kr) + C_4 K_m(kr)] \cos\phi. \tag{18c}$$

The perturbation in the external magnetic field is irrotational and derivable from another scalar function χ. Using $\nabla . \mathbf{h}^{(e)} = 0$ we get

$$\nabla^2\chi = 0. \tag{19}$$

The solution of equation (19) is

$$\chi = [C_5 I_m(kr) + C_6 K_m(kr)] \sin\phi, \tag{20}$$

and hence

$$h_r^{(e)} = k[C_5 I'_m(kr) + C_6 K'_m(kr)] \sin\phi, \tag{21a}$$

$$h_\theta^{(e)} = -(m/r)\,[C_5 I_m(kr) + C_6 K_m(kr)] \cos\phi, \tag{21b}$$

$$h_z^{(e)} = k[C_5 I_m(kr) + C_6 K_m(kr)] \cos\phi. \tag{21c}$$

To evaluate the constants C_1 to C_6 we use the following boundary conditions:

(i) Normal component of current is zero at $r = R_0$ and $r = R + a\cos\phi$.

(ii) Tangential components of H are continuous at $r = R_0$ and $r = R + a\cos\phi$.

(iii) Normal component of field is continuous at $r = R_0$ and also at $r = R + a\cos\phi$.

The condition at $r = R_0$ determines the field inside the core, assuming the thickness ϵ of the insulating sheath to be infinitesimally small.

(iv) At the bounding wall, $r = R_2$ (perfect conductor), we have $h_r^{(e)} = 0$. (22)

We thus get

$$C_1 = -aj_1 p_{m1}, \tag{23}$$

$$C_2 = aj_1 p_{m2}, \tag{24}$$

$$C_3 = -\frac{4\pi a j_1 m[p_{m1} I_m(x) - p_{m2} K_m(x)]\left(K_m(x) - \dfrac{K'_m(x_2)}{I'_m(x_2)} I_m(x)\right)}{\left(1 - \dfrac{K'_m(x_2)}{I'_m(x_2)} \dfrac{I_m(x_0)}{K_m(x_0)}\right)}, \tag{25}$$

$$C_4 = -C_3 \frac{I_m(x_0)}{K_m(x_0)}, \tag{26}$$

$$C_5 = \frac{4\pi j_1 m}{kx} \frac{\left[I_m(x) - \frac{I_m(x_0)}{K_m(x_0)} K_m(x)\right]}{\left[I_m(x) - \frac{I'_m(x_2)}{K'_m(x_2)} K_m(x)\right]}$$

$$\times \frac{[p_{m1} I_m(x) - p_{m2} K_m(x)] \left[K_m(x) - \frac{K'_m(x_2)}{I'_m(x_2)} I_m(x)\right]}{\left[\left(K'_m(x) - \frac{K'_m(x_2)}{I'_m(x_2)} I'_m(x)\right)\left(I_m(x) - \frac{I_m(x_0)}{K_m(x_0)} K_m(x)\right) - \left(K_m(x) - \frac{K'_m(x_2)}{I'_m(x_2)} I_m(x)\right)\left(I'_m(x) - \frac{I_m(x_0)}{K_m(x_0)} K'_m(x)\right)\right]}, \quad (27)$$

and
$$C_6 = -C_5 \frac{I'_m(x_2)}{K'_m(x_2)}. \quad (28)$$

Here,
$$\left. \begin{aligned} p_{m1} &= \frac{K'_m(x_0)}{I'_m(x) K'_m(x_0) - I'_m(x_0) K'_m(x)}, \\ p_{m2} &= \frac{I'_m(x_0)}{I'_m(x) K'_m(x_0) - I'_m(x_0) K'_m(x)}, \end{aligned} \right\} \quad (29)$$

where
$$x_0 = kR_0, \quad x = kR \quad \text{and} \quad x_2 = kR_2. \quad (30)$$

4. Perturbation in pressure

From equations (1) and (2) we get

$$\nabla^2 p = -4\pi j_1^2 - 8\pi j_1 \delta j_z. \quad (31)$$

Here
$$\mathbf{j} = \mathbf{j}_1 + \delta\mathbf{j}, \quad \mathbf{H} = \mathbf{H}_1 + \mathbf{h}^{(i)} \quad (32)$$

and
$$p = p_0 + \delta p, \quad \text{where} \quad \delta p_1 = a \cos\phi\, \delta p(r). \quad (32a)$$

Thus, using equations (16*c*), (23) and (24), we have

$$\nabla^2 \delta p = 8\pi k j_1^2 [p_{m1} I_m(kr) - p_{m2} K_m(kr)]. \quad (33)$$

Solution of equation (33) gives

$$\delta p = \alpha I_m(kr) + \beta K_m(kr) + 4\pi j_1^2 r [p_{m1} I'_m(kr) - p_{m2} K'_m(kr)]. \quad (34)$$

To evaluate the constants α and β occurring in equation (34) we apply the following boundary conditions

(i) Pressure should be continuous at $r = R + a\cos\phi$. (35*a*)

(ii) Equation to be satisfied at $r = R_0$ is

$$\frac{\partial \delta p}{\partial r} = \delta\mathbf{j} + \mathbf{H}_1 + \mathbf{j}_1 \times \mathbf{h}^{(i)}|_{\text{radial}} = -H_1 \delta j_z, \quad (35b)$$

since $h_\theta^{(i)} = 0$ at $r = R_0$.

Using these conditions we get

$$\alpha = \frac{\left[-2\pi j_1 R\{j_1 - (R_0^2/R^2)(j_0 - j_1)\} - 2\pi j_1 R_0\left\{j_0 - 2j_1\left(1 + \frac{m^2}{x_0^2}\right)\right\} \times [p_{m1} I_m(x_0) - p_{m2} K_m(x_0)] \frac{K_m(x)}{K'_m(x_0)}\right]}{I_m(x) - \frac{I'_m(x_0)}{K'_m(x_0)} K_m(x)}, \quad (36)$$

$$\beta = -\alpha \frac{I'_m(x_0)}{K'_m(x_0)} + \frac{2\pi j_1 R_0}{K'_m(x_0)}\left[j_0 - 2j_1\left(1 + \frac{m^2}{x_0^2}\right)\right][p_{m1} I_m(x_0) - p_{m2} K_m(x_0)].$$

5. Dispersion relation

In order to obtain the characteristic equation for ω_m we go back to equation (1), which, with the help of equation (7), gives at $r = R + a\cos\phi$,

$$\rho\omega_m^2 a \cos\phi = \frac{\partial \delta p_1}{\partial r} + j_1 h_\theta^{(i)} + H_1 \delta j_z. \quad (37)$$

Using equations (10*b*), (16*c*), (18*b*), (23)–(26), (32*a*), (34) and (36) in equation (37) we get the following dispersion relation:

$$\begin{aligned} \omega_m^2 = -\frac{4\pi j_1^2}{\rho}\Bigg[& 1 - \frac{x}{2}\left\{1 + \frac{R_0^2}{R^2} - \frac{I_c}{I}\left(1 - \frac{R_0^2}{R^2}\right)\right\} \\ & \times \left\{[p_{m1} I_m(x) - p_{m2} K_m(x)] - \frac{1}{p_{m1} I_m(x) - p_{m2} K_m(x)}\right\} \\ & - \frac{m^2}{x}(p_{m1} I_m(x) - p_{m2} K_m(x)) \left\{1 + \frac{\left(I_m(x) - \frac{I_m(x_0)}{K_m(x_0)} K_m(x)\right)\left(K_m(x) - \frac{K'_m(x_2)}{I'_m(x_2)} I_m(x)\right)}{1 - \frac{K'_m(x_2)}{I'_m(x_2)} \frac{I_m(x_0)}{K_m(x_0)}}\right\} \\ & - x_0\left\{1 - \frac{I_c}{2I}\left(\frac{R^2}{R_0^2} - 1\right) + \frac{m^2}{x_0^2}\right\}\{p_{m1} I_m(x_0) - p_{m2} K_m(x_0)\} \\ & \times \left\{\frac{K_m(x)}{K'_m(x_0)} \frac{1}{[p_{m1} I_m(x) - p_{m2} K_m(x)]} - \frac{K'_m(x)}{K'_m(x_0)}\right\}\Bigg]. \end{aligned} \quad (38)$$

6. Discussion of dispersion relation

In order to decide whether the configuration is stable or not we have to look at the sign of ω_m^2 in expression (38), which should be positive for stability. To ascertain this, the numerical calculations for ω_m^2 with $m = 0$ and 1 are done taking $R = 2R_0$ and $R_2 = 2R$.

For the case $m = 0$, it is clear from expression (38) that the external conductor has no influence on the stability of the configuration. The results for this case are presented in figure 2 for $I_c/I = 2$, 0 and -2 (curves *a*, *b*, *c* respectively). It follows from the figure that the configuration is unstable for parallel current system and even when there is no current in the core (curves *a*, *b*). The anti-parallel current system is, however, stabilizing in nature as is shown by the curve *c*. These conclusions were earlier arrived at by Tandon & Talwar, following energy arguments.

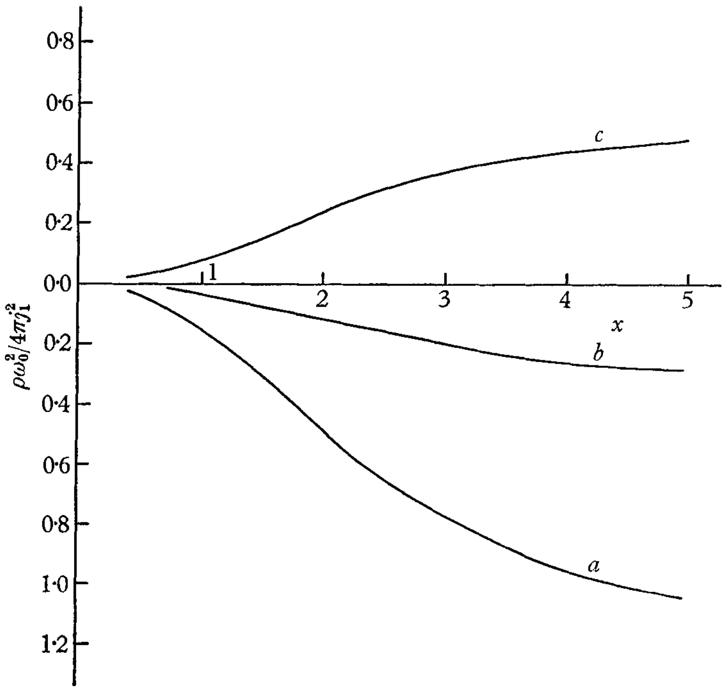

FIGURE 2. The dependence of ω_0^2 on x. Curves a, b, c refer to $I_c/I = 2, 0, -2$, respectively.

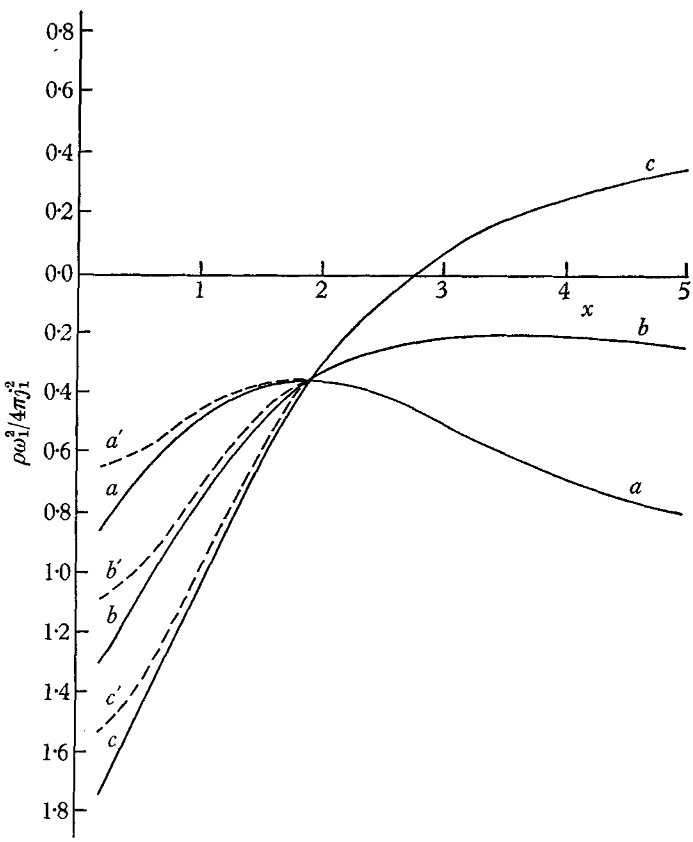

FIGURE 3. The dependence of ω_1^2 on x. Curves a, b, c refer to $I_c/I = 2, 0, -2$, respectively, in the absence of an external conductor. The effect of external conductor is shown by the dotted curves a', b', c', respectively.

Figure 3 shows the variation of the parameter ω_m^2 as a function of x $(= kR)$ in the case $m = 1$ with $I_c/I = 2$, 0 and -2 (curves a, b, c respectively referring to the case of no external conductor). It is evident from the figure that the configuration is unstable both for parallel and antiparallel current systems for kink instability. Further, the antiparallel current system is stabilizing in nature for perturbations characterized by small wavelengths whereas for long-wave perturbation the growth rate is enhanced in antiparallel current systems over parallel current systems. The effect of an external conductor is exhibited by the dotted curves which show that the external conductor has a stabilizing influence for long-wave perturbations, a result already established in problems of plasma confinement.

We conclude, therefore, that the configuration of an antiparallel current system, stable under the hydromagnetic approximation (infinite electrical conductivity), does not remain so in the limit of vanishingly small conductivity.

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Appendix

Let us assume that the region R_1 to R_2 is filled by an inviscid, incompressible fluid of zero electrical conductivity and is characterized by a pressure high enough to arrest any tendency of the detachment of the plasma from the core in the antiparallel current system. For an incompressible fluid, the equation of motion

$$\rho_2 \frac{\partial \mathbf{v}^{(e)}}{\partial t} = -\nabla \delta p^{(e)} \tag{A 1}$$

results in

$$\nabla^2 \delta p^{(e)} = 0, \tag{A 2}$$

where ρ_2 is the density of the external fluid.

Equation (2) has a solution of the form

$$\delta p^{(e)} = [A I_m(kr) + B K_m(kr)] \cos \phi. \tag{A 3}$$

Continuity of the normal component of velocity (derived from equations (1) and (3)) at $r = R + a\cos\phi$ and at $r = R_2$ yields

$$A = \frac{\omega^2 a \rho_2}{k\left[I'_m(x) - \frac{I'_m(x_2)}{K'_m(x_2)} K'_m(x)\right]} \tag{A 4a}$$

and

$$B = -A\frac{I'_m(x_2)}{K'_m(x_2)}. \tag{A 4b}$$

Proceeding exactly as before, and using the condition of continuity of pressure at the physical interface ($r = R + a\cos\phi$), we get the following modified dispersion formula:

$$\omega_m^2\left[1 - \frac{\rho_2/\rho_1}{p_{m1}I_m(x) - p_{m2}K_m(x)} \frac{K_m(x) - \frac{K'_m(x_2)}{I'_m(x_2)} I_m(x)}{K'_m(x) - \frac{K'_m(x_2)}{I'_m(x_2)} I'_m(x)}\right]$$

$$= -\frac{4\pi j_1^2}{\rho_1}\left[1 - \frac{x}{2}\left\{1 + \frac{R_0^2}{R^2} - \frac{I_c}{I}\left(1 - \frac{R_0^2}{R^2}\right)\right\}\right.$$

$$\times\left\{p_{m1}I_m(x) - p_{m2}K_m(x) - \frac{1}{p_{m1}I_m(x) - p_{m2}K_m(x)}\right\}$$

$$-\frac{m^2}{x}(p_{m1}I_m(x) - p_{m2}K_m(x))\left\{1 + \frac{\left(I_m(x) - \frac{I_m(x_0)}{K_m(x_0)}K_m(x)\right)\left(K_m(x) - \frac{K'_m(x_2)}{I'_m(x_2)}I_m(x)\right)}{1 - \frac{K'_m(x_2)}{I'_m(x_2)}\frac{I_m(x_0)}{K_m(x_0)}}\right\}$$

$$-x_0\left\{1 - \frac{I_c}{2I}\left(\frac{R^2}{R_0^2} - 1\right) + \frac{m^2}{x_0^2}\right\}\{p_{m1}I_m(x_0) - p_{m2}K_m(x_0)\}$$

$$\left.\times\left\{\frac{K_m(x)}{K'_m(x_0)}\frac{1}{p_{m1}I_m(x) - p_{m2}K_m(x)} - \frac{K'_m(x)}{K'_m(x_0)}\right\}\right]. \tag{A 5}$$

A comparison of this equation with (38) shows that the only modification to the dispersion relation due to the presence of the surrounding fluid is through the term involving ρ_2/ρ_1. The coefficient of this term can readily be seen to be always negative for $0 \leqslant x_0 \leqslant x$ and $x \leqslant x_2 \leqslant \infty$, and hence the term in parentheses with ω_m^2 is always greater than unity. Thus we conclude that the configuration of the antiparallel current system is unstable even in the presence of external gas, the growth rate of the instability being diminished, however, by its presence.